The Covariant Derivative of Tensor Densities

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In this note we want to explain how to take the covariant derivative of tensor densities. We call $\pi^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$ a tensor density of type (m,n) and weight W if

$$\frac{\pi^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}}{\sqrt{-\det \mathrm{g}}^W}$$

is a (m,n)-tensor. If π has no indices, we call it a scalar tensor density. Notice that $\sqrt{-\det \mathrm{g}}$ is the most basic scalar density there is, which we will study first.

1 The simplest case

The goal of this section is to calculate $\partial_\rho\sqrt{-\det \mathrm{g}}$ and $\nabla_\rho\sqrt{-\det \mathrm{g}}$. We will obtain this from the following fact from linear algebra:

$$\det(e^A) = e^{\mathrm{Tr}\, A},$$

for a complex-valued square matrix A. This we will not prove. We will in fact use the equivalent identity

$$\ln(\det A)) = \mathrm{Tr}(\ln A),$$

and apply it to the metric $g_{\mu\nu}$, considered as square matrix.
By differentiating the identity $\ln(\det \mathrm{g}) = \mathrm{Tr}(\ln \mathrm{g})$ with respect to x^ρ we get

$$\frac{1}{\det \mathrm{g}}\partial_\rho \det(\mathrm{g}) = \mathrm{Tr}(\mathrm{g}^{-1}\,\partial_\rho\,\mathrm{g}) = \mathrm{g}^{\mu\nu}\,\partial_\rho\,\mathrm{g}_{\mu\nu}\,.$$

Hence,

$$\begin{aligned}
\partial_\rho\sqrt{-\det \mathrm{g}} &= \partial_\rho(-\det \mathrm{g})^{\frac{1}{2}} = \frac{1}{2}(-\det \mathrm{g})^{-\frac{1}{2}}\partial_\rho(-\det \mathrm{g}) \\
&= \frac{1}{2}(-\det \mathrm{g})^{-\frac{1}{2}}\cdot -1\cdot(\det \mathrm{g})\,\mathrm{g}^{\mu\nu}\,\partial_\rho\,\mathrm{g}_{\mu\nu} \\
&= \frac{1}{2}\frac{-\det \mathrm{g}}{\sqrt{-\det \mathrm{g}}}\,\mathrm{g}^{\mu\nu}\,\partial_\rho\,\mathrm{g}_{\mu\nu} \\
&= \frac{1}{2}\sqrt{-\det \mathrm{g}}\,\mathrm{g}^{\mu\nu}\,\partial_\rho\,\mathrm{g}_{\mu\nu}
\end{aligned}$$

Recall also the definition of the Christoffel symbols:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\, \mathrm{g}^{\sigma\rho}(\partial_\mu\, \mathrm{g}_{\nu\rho} + \partial_\nu\, \mathrm{g}_\rho\, \mu - \partial_\rho\, \mathrm{g}_{\mu\nu}).$$

By contracting we get

$$\Gamma^{\mu}_{\rho\mu} = \frac{1}{2}\, \mathrm{g}^{\mu\nu}\, \partial_\rho\, \mathrm{g}_{\mu\nu}\, .$$

Combining our results we get

$$\partial_\rho \sqrt{-\det \mathrm{g}} = \frac{1}{2}\sqrt{-\det \mathrm{g}}\, \mathrm{g}^{\mu\nu}\, \partial_\rho\, \mathrm{g}_{\mu\nu} = \sqrt{-\det \mathrm{g}}\; \Gamma^{\mu}_{\rho\mu}.$$

We can replace the partial derivate by the covariant derivate ∇ in the first 2 terms, since for these steps we have used only standard properties of derivaties (chain rule/ product rule etc.). Also notice that by metric compatibily of ∇ we have $\nabla_\rho\, \mathrm{g}_{\mu\nu} = 0$, hence it follows

$$\nabla_\rho \sqrt{-\det \mathrm{g}} = \frac{1}{2}\sqrt{-\det \mathrm{g}}\, \mathrm{g}^{\mu\nu}\, \nabla_\rho\, \mathrm{g}_{\mu\nu} = 0.$$

2 Tensor Densities

From now on we will write $\sqrt{-\mathrm{g}}$ instead of $\sqrt{-\det \mathrm{g}}$ for brevity. For simplicity in the following we assume π to be a scalar tensor density of weight 1. We now want to take the covariant derivative of π, i.e we want to compute $\nabla_\rho \pi$. Since $\frac{\pi}{\sqrt{-\det \mathrm{g}}}$ is a scalar we have

$$\nabla_\rho(\frac{\pi}{\sqrt{-\mathrm{g}}}) = \partial_\rho(\frac{\pi}{\sqrt{-\mathrm{g}}}).$$

By the product rule and the fact that ∇ is metric-compatible we find

$$\nabla_\rho \pi = \sqrt{-\mathrm{g}}\left((\partial_\rho \pi)\cdot \frac{1}{\sqrt{-\mathrm{g}}} - \pi \frac{\partial_\rho \sqrt{-\mathrm{g}}}{(\sqrt{-\mathrm{g}})^2}\right).$$

And by our previous result, we conclude

$$\nabla_\rho \pi = \partial_\rho \pi - \Gamma^{\mu}_{\rho\mu}\pi.$$

This result can easily be extended to arbitrary tensor densities. Indeed, let for example $\pi^{\mu\nu}$ be a two-tensor density of weight 1. We can just calculate the covariant derivate of the tensor $\frac{\pi^{\mu\nu}}{\sqrt{-\mathrm{g}}}$:

$$\nabla_\rho\left(\frac{\pi^{\mu\nu}}{\sqrt{-\mathrm{g}}}\right) = \partial_\rho\left(\frac{\pi^{\mu\nu}}{\sqrt{-\mathrm{g}}}\right) + \Gamma^{\mu}_{\rho\kappa}\frac{\pi^{\mu\nu}}{\sqrt{-\mathrm{g}}} + \Gamma^{\nu}_{\rho\kappa}\frac{\pi^{\mu\nu}}{\sqrt{-\mathrm{g}}}$$

And by calculating the partial derivative on the right hand side as before, we get

$$\nabla_\rho \pi^{\mu\nu} = \sqrt{-\mathrm{g}}\cdot \nabla_\rho\left(\frac{\pi^{\mu\nu}}{\sqrt{-\mathrm{g}}}\right) = \partial_\rho \pi^{\mu\nu} + \Gamma^{\mu}_{\rho\kappa}\pi^{\kappa\nu} + \Gamma^{\nu}_{\rho\kappa}\pi^{\mu\kappa} - \Gamma^{\kappa}_{\rho\kappa}\pi^{\mu\nu}.$$